## Section One: Calculator-free (40 Marks)

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

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### Question 1 (4 marks)

Find the minimum and maximum values of $f(x) = 2x^3 - 3x^2 - 12x + 27$ over the interval $-3 \le x \le 3$.

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1)$$
$$x = -1 \text{ or } 2 \quad \checkmark$$

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 27$$
$$= -2 - 3 + 12 + 27 = \underline{34}$$
$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 27$$
$$= 16 - 12 - 24 + 27 = \underline{7}$$
$$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 27$$
$$= 54 - 27 - 36 + 27 = \underline{18}$$
$$f(-3) = 2(-3)^3 - 3(-3)^2 - 12(-3) + 27$$
$$= -54 - 27 + 36 + 27 = \underline{-18}$$

$\checkmark\checkmark\checkmark$

−1 / mistake
−2 if endpoints not tested.

Max value 34
min value −18.

**Question 2** **(5 marks)**

Find $\frac{dy}{dx}$ in terms of $x$ for each of the following.

(a) $y = x(1+2e^{3x})$ (2 marks)

$$\frac{dy}{dx} = (1+2e^{3x}).1 + x(6e^{3x}) \checkmark \checkmark$$
$$= 1 + 2e^{3x} + 6xe^{3x}$$

−1 if mistake in simplifying

(b) $y = \int_1^x t^2 + t - 1 \; dt$ (1 mark)

$$\frac{dy}{dx} = \frac{d}{dx}\int_1^x t^2 + t - 1 . dt$$
$$= x^2 + x - 1 \checkmark$$

(c) $y = z^3 - z$ and $z = x^2 - 9$ (2 marks)

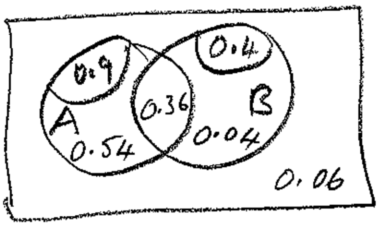
$$\frac{dy}{dz} = 3z^2 - 1 \qquad \frac{dz}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$
$$= (3z^2 - 1)\,2x \checkmark$$
$$= 2x(3(x^2-9)^2 - 1)$$
$$= 6x(x^2-9)^2 - 2x \checkmark$$

**Question 3** **(5 marks)**

Two independent events A and B are such that $P(A) = 0.9$ and $P(B) = 0.4$.

(a) Find $P(\overline{A \cup B})$. (2 marks)




$P(A \cap B) = P(A) \times P(B)$
$= 0.36$ ✓

$P(\overline{A \cup B}) = 0.06$ ✓

(b) Find $P(\,\bar{B} \mid \bar{A} \cup B\,)$. (1 mark)

$\dfrac{P(\bar{B} \cap \bar{A} \cup B)}{P(\bar{A} \cup B)} = \dfrac{0.06}{0.46}$

$= \dfrac{6}{46}$ or $\dfrac{3}{23}$ ✓

(c) Show that $\bar{A}$ and $\bar{B}$ are also independent. (2 marks)

$P(\bar{A}) = 0.1$ $P(\bar{B}) = 0.6$

$P(\bar{A} \cap \bar{B}) = 0.06$ from Venn diagram ✓

NB $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

Hence
$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \times P(\bar{B})$
$0.06 = 0.1 \times 0.6$ ✓
$\therefore$ Independent

**Question 4** **(7 marks)**

Two functions are defined as $f(x)=\sqrt{x-1}$ and $g(x)=\dfrac{1}{x-1}$.

(a) Evaluate $g\circ f\left(\dfrac{13}{9}\right)$. (2 marks)

$$g\circ f(x)=\frac{1}{\sqrt{x-1}-1}$$

$$g\circ f\left(\frac{13}{9}\right)=\frac{1}{\sqrt{\frac{13}{9}-1}-1} \checkmark$$

$$=\frac{1}{\sqrt{\frac{4}{9}}-1}$$

$$=\frac{1}{\frac{2}{3}-1}=\frac{1}{-\frac{1}{3}}=-3 \checkmark$$

(b) Find in simplified form $g\circ g(x)$. (2 marks)

$$g\circ g(x)=\frac{1}{\frac{1}{x-1}-1} \checkmark$$

$$=\frac{1}{\frac{1-(x-1)}{x-1}}$$

$$=\frac{1}{\frac{2-x}{x-1}}=\frac{x-1}{2-x} \checkmark$$

(c) Determine the domain of $f(g(x))$. (3 marks)

$$f(g(x))=\sqrt{\frac{1}{x-1}-1}$$

$$\frac{1}{x-1}-1\geq 0$$

$$\frac{1}{x-1}\geq 1$$

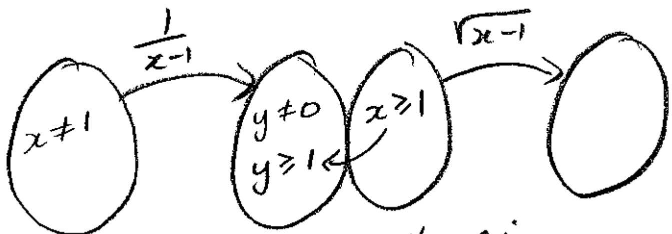
$x>1$ $\quad 1\geq x+1$

$2\geq x$

$x$ cannot be $<1$ because $\frac{1}{x-1}$ cannot be negative

$\therefore \quad 1<x\leq 2$

OR.




Need to change domain

$\frac{1}{x-1}\geq 1 \quad \therefore \quad 1<x\leq 2$

OR $\quad \frac{1}{x-1}-1\geq 0$

$$\frac{1-(x-1)}{x-1}\geq 0$$

$$\frac{2-x}{x-1}\geq 0$$

Hence $1<x\leq 2$

**Question 5** **(4 marks)**

Solve the system of equations
$$c + 2a = 3 + 4b$$
$$a + 2b + 2c = 4$$
$$5a + 3c = 5 + 2b$$

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & -4 & 1 & 3 \\ 5 & -2 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -8 & -3 & -5 \\ 0 & -12 & -7 & -15 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & +8 & +3 & +5 \\ 0 & 0 & 5 & 15 \end{bmatrix} \begin{array}{l} \\ 3R_2 - 2R_3 \\ \\ \end{array}$$

$5c = 15 \quad \therefore \ c = 3$

$8b + 9 = 5 \quad \therefore \ b = -\frac{1}{2}$

$a - 1 + 6 = 4 \quad \therefore \ a = -1$

(-1 overall if no +c)



**Question 6** **(5 marks)**

(a) Determine $\int \frac{2e^{-0.2y}}{5}\, dy$. (1 mark)

$= -2\int \left(-\frac{2}{10}\right) e^{-0.2y} . dy$

$= -2e^{-0.2y} + c$ ✓

(b) Determine $\int (t-1)\left(1-2t+t^2\right)^3 dt$. (2 marks)

$$\left.\begin{aligned} &\frac{1}{2}\int \left(1-2t+t^3\right)^3 (2t-2)\, dt \\ &= \frac{1}{2}\,\frac{\left(1-2t+t^3\right)^4}{4} \end{aligned}\right\} ✓$$

$= \frac{\left(1-2t+t^3\right)^4}{8} + c$ ✓

(c) Evaluate $\int_1^6 \frac{3}{x^2}\, dx$. (2 marks)

$\int_1^6 3x^{-2} . dx = \left[\frac{3x^{-1}}{-1}\right]_1^6$ ✓

$= \left.-\frac{3}{x}\right]_1^6$

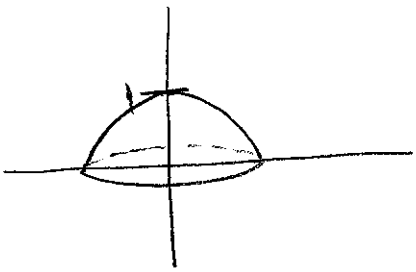
$= -\frac{1}{2} - (-3)$

$= 2\frac{1}{2}$ ✓

**Question 7** **(4 marks)**

The region in the first quadrant bounded by $x=0$, $y=0$ and $y=1-\frac{x^2}{9}$ is rotated 360° about the $y$-axis. If $x$ and $y$ are distances measured in centimetres, find the volume of the solid formed.

$y = -\frac{x^2}{9} + 1$



$V = \pi \int x^2 \, dy$

$V = \pi \int_0^1 -9(y-1) \, dy$

$V = \pi \int_0^1 9 - 9y \, . \, dy$

$= \pi \left[ 9y - \frac{9y^2}{2} \right]_0^1$

$= \pi \left( 9 - \frac{9}{2} \right)$

$= \frac{9\pi}{2} \text{ cm}^3$

**Question 8** **(6 marks)**

The variables $k$ and $m$ are both integers such that $m^2 + 3 = 2k$.

(a) Use counter-examples to disprove any two of the three conjectures listed below. (2 marks)

- $m$ can be any even integer.

Let $m$ be 2  $2^2 + 3 = 7$
$2k = 7$ means $k = 3.5$ (Not an integer)
$\therefore$ Statement false. ✓

- $m$ can be any odd integer.

Statement always true.

- $m$ must be a positive odd integer.

Let $m = -3$  $m^2 + 3 = 9 + 3 = 12$
$2k = 12 \therefore k = 6$ ✓

Statement can be true for negative integers $\therefore$ False.

(b) Using the fact that any odd integer can be written in the form $2n + 1$ or otherwise, prove that $k$ is always the sum of three square numbers. (4 marks)

Let $m$ be any odd integer $2n+1$

$$
\begin{aligned}
m^2 + 3 &= (2n+1)^2 + 3 \\
&= 4n^2 + 4n + 1 + 3 \\
&= 4n^2 + 4n + 4 \\
&= 2k \\
\therefore k &= 2n^2 + 2n + 2 \\
&= n^2 + \underbrace{n^2 + 2n + 1} + 1 \\
&= n^2 + (n+1)^2 + 1
\end{aligned}
$$

QED

End of questions